Filter Banks - V

Paraunitary Perfect Reconstruction Filter Banks

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Paraunitary Perfect Reconstruction Filter Banks.

Introduction:

Analysis and synthesis filter banks of M channel maximally decimated filter bank can be expressed in terms of polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$. Such a filter bank with **FIR** filters has 'perfect reconstruction' property iff $\mathbf{E}(z)$ is just a delay. i.e.

$$\det \mathbf{E}(z) = \alpha\, z^{-K} .$$

We shall discuss **P**erfect **R**econstruction filter banks in which the polyphase matrix $\mathbf{E}(z)$ satisfies a special property called the ***lossless*** or ***Paraunitary*** property

- Synthesis filter and analysis filters have the same length
- This property is basic to the generation of the "Orthonormal Wavelet basis "

Paraunitary Property

In our earlier discussions, the analysis bank is described by an $M \times 1$ transfer matrix $\mathbf{h}(z)$ and the synthesis filter by $1 \times M$ transfer matrix $\mathbf{f}^T(z)$ which are expressed in terms of polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ as:

$$\mathbf{h}(z) = \mathbf{E}(z^M)\ .\ \mathbf{e}(z) \qquad \mathbf{f}^T(z) = z^{-(M-1)}\, \mathbf{e}^{\sim}(z)\, \mathbf{R}(z^M) \qquad \text{…. (1)}$$

Lossless transfer Matrix:

A $p \times r$ causal matrix $\mathbf{H}(z)$ is said to be lossless if

- each entry $H_{km}(z)$ is stable
- $\mathbf{H}(e^{jw})$ is unitary, that is,

$$\mathrm{H}^{\blacklozenge}(e^{jw}).\ \mathrm{H}(e^{jw}) = d\mathrm{I}_r \qquad (\ \mathrm{H}^{\blacklozenge}(e^{jw}) \text{ is transpose-conjugate of } \mathrm{H}(e^{jw}). \qquad \text{…. (2)}$$

"$\mathbf{H}(z)$ is lossless" is equivalent to " the LTI system with transfer function $\mathbf{H}(z)$ is lossless."

Paraunitary Property

Equation (2) is called Unitary property. Thus **H**(z) is unitary on the unit circle in the Z plane.

In order to satisfy (2), $p \geq r$. The subscript 'r' in I_r means that it is a r x r matrix.

Paraunitary Property:

For rational transfer functions, (2) implies that:

$$\tilde{H}(z).H(z) = dI, \quad \text{for all } z \qquad \text{….. } \mathbf{(3)}$$

This is termed as *paraunitary property.*

(Note: $\tilde{H}(z)$ is the complex conjugate of H(z)

i.e. $\tilde{H}(z) = H_*(z^{-1})$

Ex. : - Let $H(z) = (a + bz^{-1})$, then

$$\tilde{H}(z) = a^* + b^* z \qquad)$$

So, for a causal system to be lossless, it is sufficient to prove

* stability * paraunitariness.

Paraunitary property

Observations:

1. If $\mathbf{H}(z)$ is square and lossless, then $\tilde{H}(z)$ is paraunitary but not lossless (unless it is a constant).

2. "Lossless" and "Paraunitariness" are used interchangeably.

Normalized systems: If a lossless system has $d = 1$, then we say it as *normalized-lossless.*

Square matrices:

For square matrices, equation (2) implies that

$$H^{-1}(z) = \tilde{H}(z) / d$$

i.e. the inverse of the matrix can be obtained by use of 'tilde' operation.

In this case, every row is power complementary, and any pair of rows is orthogonal since

$$\tilde{H}(z).H(z) = H(z).\tilde{H}(z) = dI$$

Properties of Paraunitary Systems.

(Note: Power complimentary transfer functions:

$H_0(z), H_1(z)$ are said to be power complimentary if

$$|H_0(e^{j?})|^2 + |H_1(e^{j?})|^2 = c^2 \quad \text{for all} ?$$)

Some properties of Paraunitary Systems:

1. Determinant is allpass. For a square matrix, | **H**(z)| is all pass, in particular, if **H**(z) is FIR then, | **H**(z) | is a delay i.e.

$$\det \mathbf{H}(z) = a\, z^{-K}, \qquad K \geq 0, \quad a\ \sigma\ 0$$

2. Power Complimentary Property. For a M x 1 transfer matrix **h**(z) = [$H_0(z)$ …… $H_{M-1}(z)$], then

$$\sum_{k=0}^{M-1} |H_k(e^{j?})|^2 = c \qquad \text{for all} ?$$

….. (**4**)

3.*Submatrices of paraunitary* **H**(z). Every column of a paraunitary transfer matrix is itself paraunitary.

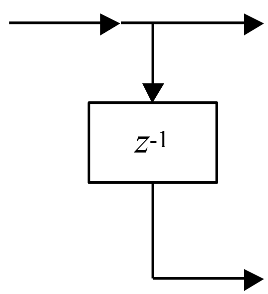
Examples:

1. $\mathbf{K}(z) = \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix}$ then we have

$$\tilde{\Lambda}(z).\Lambda(z) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} = I$$

Therefore, $\mathbf{K}(z)$ is paraunitary.

2. The system in the adjacent figure has a transfer matrix



$$e(z) = \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

$$\tilde{e}(z) = \begin{bmatrix} 1 & z \end{bmatrix}$$

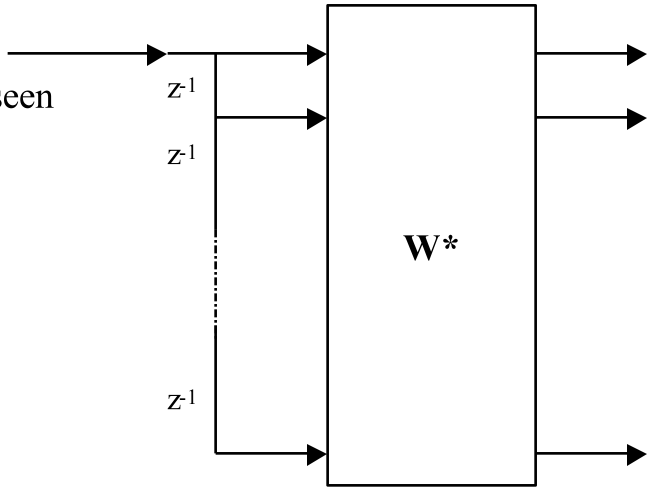
Therefore, $\tilde{e}(z).e(z) = \begin{bmatrix} 1 & z \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} = 2$

So, $\mathbf{e}(z)$ is paraunitary !

Examples

3. Paraunitary Filter banks:

Consider the system in the figure, which is a cascade of two systems with transfer matrices $\mathbf{e}(z)$ and $\mathbf{W^*}$ respectively



W is M x M DFT matrix which is unitary, we have already seen that $\mathbf{e}(z)$ is also paraunitary in previous example, i.e.

$$\tilde{e}(z).e(z) = M$$

Thus, Overall transfer matrix is also paraunitary

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ . \\ . \\ . \\ \\ H_{M-1}(z) \end{bmatrix} = W^* e(z)$$

Filter Bank Properties

From the previous discussion, the paraunitary property implies

$$\tilde{E}(z)E(z) = d\,I \text{ , that is, } E^{-1}(z) = \tilde{E}(z)/d \text{ for all } z$$

So we choose **R**(z) as

$$R(z) = cz^{-K}\tilde{E}(z) \qquad \text{….. (5)}$$

Note: Positive K ensures that **R**(z) is causal.

Stability:

If the analysis filters are stable and IIR, then choice of R(z) as per equation (3) results in unstable filters !

So, we cannot build useful Perfect Reconstruction Systems with IIR lossless **E**(z) !!
Hence we restrict our attention to **FIR** filters.

Properties

Properties:

1. Relation between Analysis and Synthesis Filters:

Substituting equation (5) in equation (1) for synthesis filters, we obtain

$$\mathbf{f}^T(z) = z^{-(M-1)}\,\mathbf{e}^{\sim}(z)\,\mathbf{R}(z^M)$$

$$= c\,z^{-(M-1+MK)}\,\mathbf{e}^{\sim}(z)\;\mathbf{E}^{\sim}(z^M)$$

$$= c\,z^{-(M-1+MK)}\,\mathbf{h}^{\sim}(z). \qquad \text{Let } L = M\text{-}1+MK$$

$$F_k(z) = c\,z^{-L}\;\widetilde{H}_k(z) \qquad \text{..... (}\mathbf{6}\text{)}$$

In time domain, it can be expressed as:

$$f_k(n) = c\,h^*_k(L - n) \quad , \qquad 0 < k < M\text{-}1$$

In frequency domain, it implies that

$$|F_k(z)| = |\,c\,|\;|\,H_k(e^{jw})|$$

i.e. the magnitude responses of $F_k(z)$ and $H_k(z)$ are exactly the same (with a scale factor c)

Properties

Theorem:

Consider a maximally decimated QMF bank with causal FIR analysis filters $H_k(z)$, and let $\mathbf{E}(z)$ be the polyphase matrix for the analysis filters. Then,

1. $\mathbf{E}(z)$ is lossless (that is, paraunitary)

2. The synthesis filters are given by $f_k(n) = c\, h^*_k(L - n), \quad 0 \leq k \leq M\text{-}1$

3. The system has perfect reconstruction property.

2. Power Complimentary property:

Consider the vector of analysis filters $\mathbf{h}(z) = \mathbf{E}(z^M)\, \mathbf{e}(z)$. $\mathbf{h}(z)$ is paraunitary which implies that analysis filters $H_k(z)$ are power complimentary i.e.

$$\sum_{k=0}^{M-1} \left| H_k(e^{j?}) \right|^2 = \text{positive constant}$$

Properties

3. AC matrix is paraunitary if and only if $\mathbf{E}(z)$ is Paraunitary.

4. Relation to Mth band filters

If $\mathbf{E}(z)$ is Paraunitary, the each analysis filter $H_k(z)$, is a spectral factor of a (Zero phase) Mth band filter. The filter $G_k(z)$, defined as:

$$G_k(z) \cong \tilde{H}_k(z).H_k(z)$$

is an Mth band filter.

Two Channel FIR paraunitary QMF Banks

Consider a two channel QMF filter bank with causal FIR filters given by

$$H_0(z) = \sum_{n=0}^{N} h_0(n) z^{-n} \qquad\qquad H_1(z) = \sum_{n=0}^{N} h_1(n) z^{-n}$$

The alias-component matrix (AC) is given by:

$$H(z) = \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}$$

Paraunitariness of **H**(z) implies that $\widetilde{H}_k(z).H_k(z) = ⌡\mathrm{I}$, where $⌡ = 2d$.

From this, we obtain:

$$\widetilde{H}_0(z).H_0(z) + \widetilde{H}_0(\text{-}z).H_0(\text{-}z) = ⌡ \qquad \text{..... (7 a)}$$

$$\widetilde{H}_1(z).H_1(z) + \widetilde{H}_1(\text{-}z).H_1(\text{-}z) = ⌡ \qquad \text{..... (7 b)}$$

$$\widetilde{H}_0(z).H_1(z) + \widetilde{H}_0(\text{-}z).H_1(\text{-}z) = 0 \qquad \text{..... (7 c)}$$

Two Channel FIR paraunitary QMF Banks

The above equations imply that

$$|\widetilde{H}_0(z).H_0(z)|_{\downarrow 2} = 0.5 \quad |\widetilde{H}_1(z).H_1(z)|_{\downarrow 2} = 0.5 \quad |\widetilde{H}_0(z).H_1(z)|_{\downarrow 2} = 0 \qquad \text{.....(8)}$$

From this, we can say that:

- $\widetilde{H}_0(z).H_0(z)$ is a half-band filter, i.e. $H_0(z)$ is *power symmetric*.
- Order of $\mathbf{H}_0(z)$ is necessarily odd, $N = 2J + 1$

Relation between the Two Analysis Filters:

From equation (**7 b**) we have that

$$\frac{H_1(z)}{H_0(z)} = \frac{-\widetilde{H}_0(-z)}{\widetilde{H}_1(-z)}$$

Two Channel FIR paraunitary QMF Banks

From equation (**7 a**) we have that $\tilde{H}_0(z).H_0(z) + \tilde{H}_0(\text{-}z).H_0(\text{-}z) = J$

which implies that there are no common factors between $H_0(z)$ and $H_1(z)$ (since right hand side is a constant)

Hence we conclude that

$$H_1(z) = cz^{-L}\tilde{H}_0(\text{-}z) \qquad \text{..... (} \mathbf{9} \text{)}$$

This is equivalent to in frequency domain as:

$$|H_1(e^{j\xi})| = |H_0(-e^{j\xi})| = |H_0(e^{j(\xi-\nu)})|$$

i.e. the magnitude response of $H_1(z)$ is obtained by shifting that of $H_0(z)$by π.
For a real coefficient case, this means that if $H_0(z)$ is low-pass then $H_1(z)$ is high-pass both filters have the same ripple sizes, and same transition band-widths.

Two Channel FIR paraunitary QMF Banks

Design of Perfect Reconstruction QMF bank:

- First design a zero-phase half-band filter $H(z)$ with $H(e^{jw}) \geq 0$.
- Compute the spectral factor $H_0(z)$ (see section 3.2.5 or appendix D of text) which gives one of the analysis filters with order $N = 2J + 1$.
- Obtain the other analysis filter $H_1(z)$ and the two synthesis filters $F_0(z)$, $F_1(z)$ as:

$$H_1(z) = -z^{-N}\widetilde{H}_0(-z), \;\; F_0(z) = z^{-N}\widetilde{H}_0(z), \;\; and\; F_1(z) = z^{-N}\widetilde{H}_1(z) \qquad .. (\mathbf{10})$$

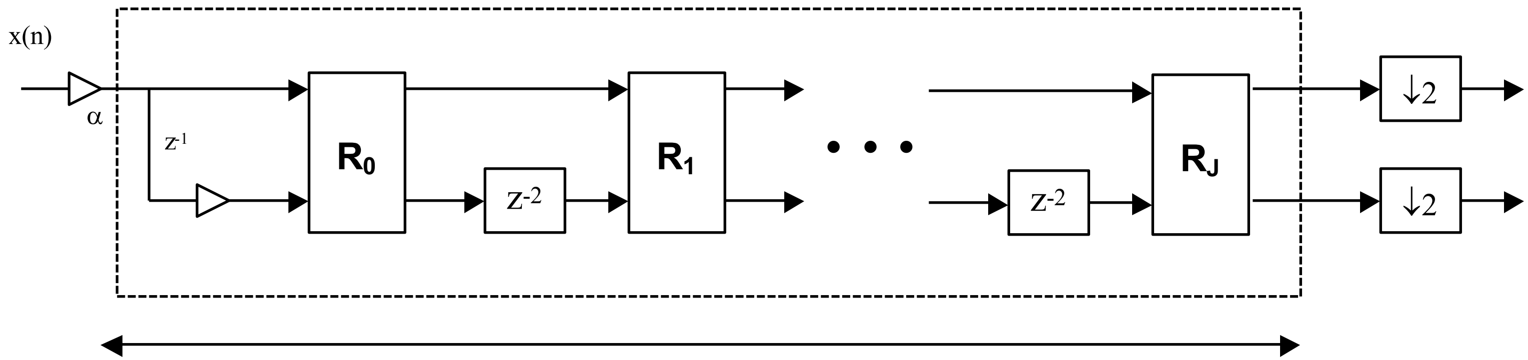
Equivalently, the above expression can be written as: .. (**11**)

$$h_1(n) = (-1)^n h_0^*(N - n) \qquad f_0(n) = h_0^*(N - n), \text{ and } \quad f_1(n) = h_1^*(N - n)$$

<u>Two Channel FIR paraunitary QMF Lattice</u>

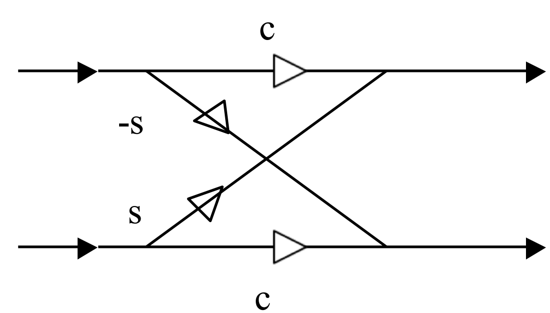
FIR Two channel QMF bank with real coefficients:

Consider the following cascaded structure.



R_m is called Givens rotation, transfer matrix defined and implemented as:

$$R_m(z) = \begin{bmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{bmatrix} \quad \theta_m \text{ is real}$$



* R_m is unitary

c denotes cos(θm) and s denotes sin(θm)

Two Channel FIR paraunitary QMF Lattice

Any 2x2 real coefficient (causal, FIR) paraunitary matrix can be factored as:

$$E(z) = aR_J.\Lambda(z)R_{J-1}....\Lambda(z)R_0\begin{bmatrix}1 & 0\\ 0 & \pm 1\end{bmatrix}$$

$$\text{where} \quad \Lambda(z) = \begin{bmatrix}1 & 0\\ 0 & z^{-1}\end{bmatrix}, \quad \text{a is a positive scalar}$$

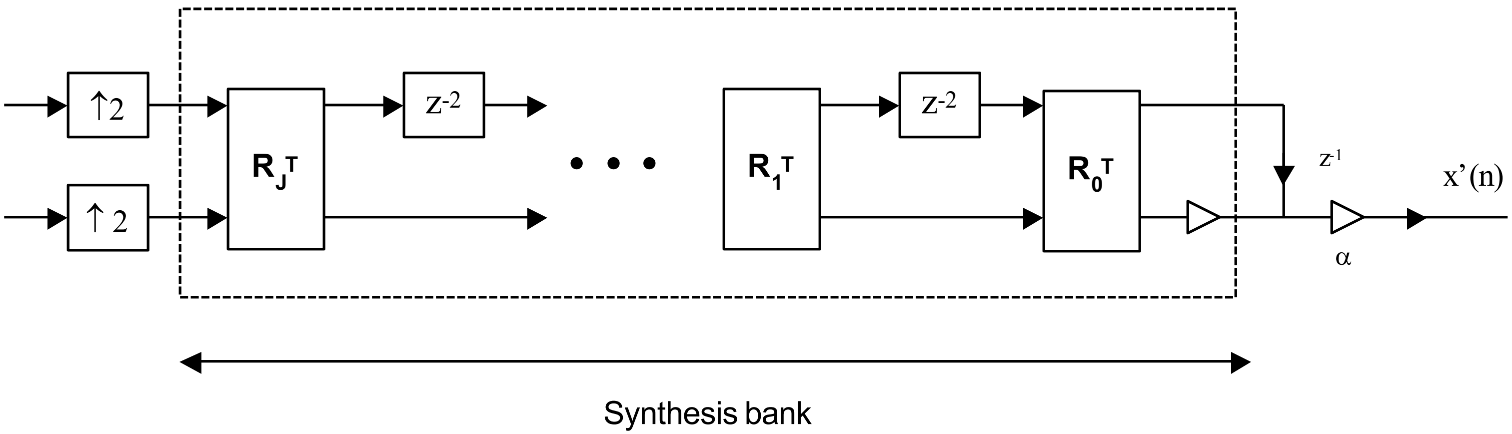
Synthesis bank which would result in perfect reconstruction is given by applying equation (5):

$$R(z) = a\begin{bmatrix}1 & 0\\ 0 & \pm 1\end{bmatrix}R_0^T.\Gamma(z).......R^T_{J-1}.\Gamma(z)R_J$$

$$where, \Gamma(z) = \begin{bmatrix}z^{-1} & 0\\ 0 & 1\end{bmatrix}$$

Two Channel FIR paraunitary QMF Lattice

Lattice structure for synthesis filter bank:



Analysis and Synthesis filters have order $N = 2J + 1$.

Two Channel FIR paraunitary QMF Lattice

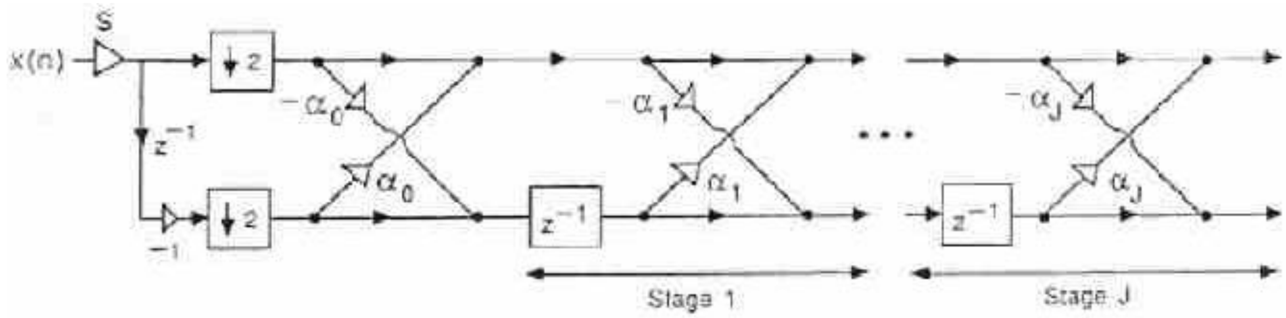
In a more efficient lattice structure, the rotation matrix $\mathbf{R}_m$ can be written as:

$$R_m = \cos?_m \begin{bmatrix} 1 & a_m \\ -a_m & 1 \end{bmatrix} \quad \text{if } \cos?_m \neq 0$$

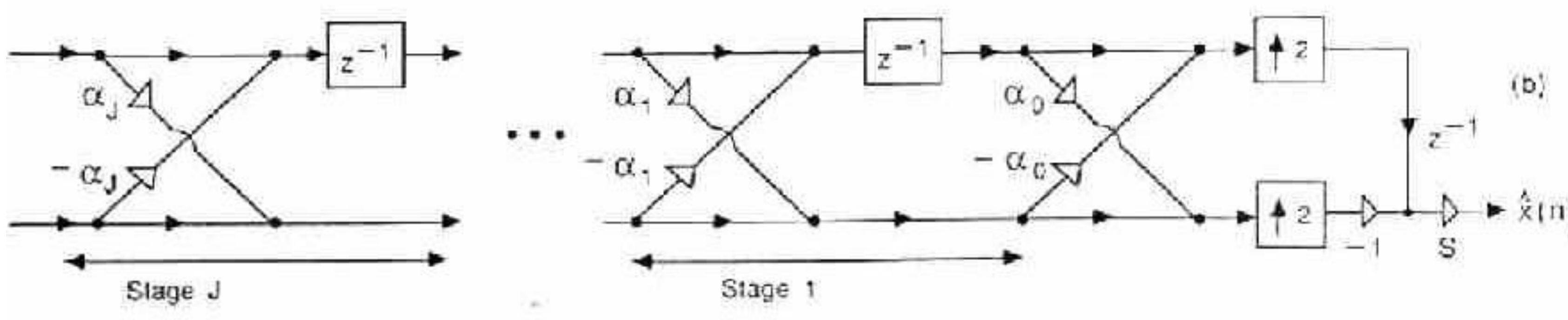
$$R_m = \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{otherwise.}$$

Lattice structure can now be redrawn as shown in the following figure with $S = \alpha \prod_m \cos?_m$

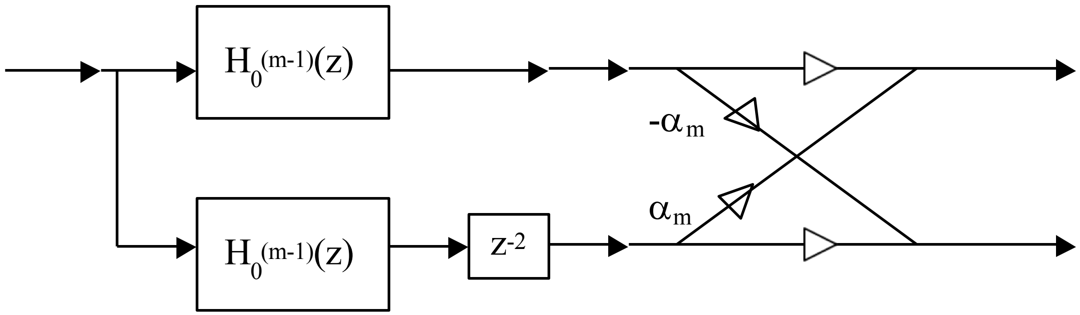
Two Channel FIR paraunitary QMF Lattice



Analysis Bank



Synthesis Bank



Schematic for the m-th stage

The *m*th stage filters can be obtained from m-1 th stage filters as:

$$H_0^{(m)}(z) = H_0^{(m-1)}(z) + a_m z^{-2} H_1^{(m-1)}(z),$$
$$H_1^{(m)}(z) = -a_m H_0^{(m-1)}(z) + z^{-2} H_1^{(m-1)}(z) \quad \text{.. (\textbf{12})}$$

The coefficients αm are calculated by inverting the above equations to obtain:

$$(1 + a_m^{\ 2}) H_0^{(m-1)}(z) = H_0^{(m)}(z) - a_m H_1^{(m)}(z),$$
$$(1 + a_m^{\ 2}) z^{-2} H_1^{(m-1)}(z) = a_m H_0^{(m)}(z) + H_1^{(m)}(z) \quad \text{.. (\textbf{13})}$$

Derivation of 13 from 12:

From 12

$$H_0^{(m)}(z) = H_0^{(m-1)}(z) + a_m z^{-2} H_1^{(m-1)}(z),$$
$$H_1^{(m)}(z) = -a_m H_0^{(m-1)}(z) + z^{-2} H_1^{(m-1)}(z)$$

which can be written as:

$$\begin{bmatrix} H_0^{(m)}(z) \\ H_1^{(m)}(z) \end{bmatrix} = \begin{bmatrix} 1 & a_m z^{-2} \\ -a_m & z^{-2} \end{bmatrix} \begin{bmatrix} H_0^{(m-1)}(z) \\ H_1^{(m-1)}(z) \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} H_0^{(m-1)}(z) \\ H_1^{(m-1)}(z) \end{bmatrix} = \begin{bmatrix} 1 & a_m z^{-2} \\ -a_m & z^{-2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} H_0^{(m)}(z) \\ H_1^{(m)}(z) \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} H_0^{(m)}(z) \\ H_1^{(m)}(z) \end{bmatrix} = \frac{1}{a_m^2 z^{-2} + z^{-2}} \begin{bmatrix} z^{-2} & -a_m z^{-2} \\ a_m & 1 \end{bmatrix} \begin{bmatrix} H_0^{(m-1)}(z) \\ H_1^{(m-1)}(z) \end{bmatrix}$$

$$\Rightarrow \quad (1 + a_m^2)\, z^{-2} \begin{bmatrix} H_0^{(m)}(z) \\ H_1^{(m)}(z) \end{bmatrix} = \begin{bmatrix} z^{-2} & -a_m z^{-2} \\ a_m & 1 \end{bmatrix} \begin{bmatrix} H_0^{(m-1)}(z) \\ H_1^{(m-1)}(z) \end{bmatrix}$$

From which it follows that:

$$(1 + a_m^2) H_0^{(m-1)}(z) = H_0^{(m)}(z) - a_m H_1^{(m)}(z),$$
$$(1 + a_m^2) z^{-2} H_1^{(m-1)}(z) = a_m H_0^{(m)}(z) + H_1^{(m)}(z)$$

Two Channel FIR paraunitary QMF Lattice

Properties of Paraunitary QMF Lattice:

The properties of the QMF lattice are almost similar to that dis cussed in previous section(s).

Completeness:

- Every two channel (real coefficient, FIR) paraunitary QMF bank can be represented using the above lattice structure.
- We can always define $H_1(z) = -z^{-N} H_0(-z^{-1})$ and implement the analysis bank using the above lattice, given a real coefficient power symmetric FIR filter $H_0(z)$

Complexity of Paraunitary QMF lattice:

The total number of multipliers required to implement the lattic e sections in the analysis is equal to $2(J + 1) + 2$.

Each of these operates at half the input sampling rate, so that we have an average of $J + 2$ MPU's.

Therefore, MPU's to implement the lattice sections in analysis bank $= J + 2 = 0.5(N + 3)$.

Each lattice section requires two additions, so $J + 1$ sections require $2(J + 1)$ additions and each operate at half the input sampling rate.

Therefore, total number of APU's $= (J + 1) = 0.5(N + 1)$.

Synthesis bank has the same complexity.

Thus, lattice structure is more efficient, requiring only **half** as many MPU's as the direct form !